Pressure Releasing Policy in Traffic Signal Control with Finite Queue Capacities

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Abstract—This paper deals with traffic signal control with finite queue capacities in a discrete-time and stochastic setting. A so-called “pressure releasing policy” (PRP) is introduced to optimally release traffic pressure at every time slot, where the traffic pressure at each intersection incorporates knowledge of turning ratios and information of neighboring and ingress queues. PRP does not require knowledge of arrival rates. Moreover, it employs a set of weights satisfying a given condition to handle downstream queue spillover, and an algorithm is provided to generate one possible set of weights. Define the throughput region as the closure of the set of all arrival rate vectors that can be stably supported over the network under the assumption on infinite queue capacities. It is shown that PRP under finite queue capacities can still achieve the closed-loop stability with a reduction on the throughput region. The reduction is a function of weights and internal queue capacities, and PRP with finite but sufficiently large internal queue capacities can be arbitrarily close to recovering the throughput region.

I. INTRODUCTION

It is well known that traffic congestion causes plenty of air pollution and efficiency loss. Due to budget constraint and space limitation, it is difficult to enlarge road capacity. With the continued growth of travel demand, traffic signal control becomes increasingly important to modern cities, which aims at maximizing the efficiency (e.g., throughput) of road networks without requiring to increase road capacity. See [1] for reviews on traffic signal control. The advance of communication, control and information technology also makes it possible to create intelligent traffic signal control systems, e.g., SCATS [2], [3], SCOOT [4], [5], TUC [6], OPAC [7], etc. However, none of the above traffic signal control systems can ensure the closed-loop stability, which gives rise to uncertainty about their performance especially in the heavy/overloaded traffic regime.

The most pertinent work to this paper includes [8]–[10] and [11], [12]. Adapted from the joint routing and scheduling algorithm proposed in the area of wireless communication [13], [14], back-pressure algorithm [8] and max-pressure algorithm [9], [10] are introduced to traffic signal control. Both algorithms are distributed and can achieve maximum throughput region under the assumption on infinite queue capacities/storage spaces. Stochastic models are adopted in [8] and [10] with different stability notions, while a deterministic model is used in [9]. The throughput region is defined in terms of arrival rates that can be stably supported over the network. However, the length of any internal queue within a traffic network should be physically constrained by the length of the corresponding link. Therefore, the assumption on infinite queue capacities is violated, and it is direct to show that under finite queue capacities, back-pressure algorithm [8] and max-pressure algorithm [9], [10] cannot maintain the guarantee on maximum throughput region.

On the other hand, [11], [12] extend the results on the case with infinite buffer sizes in [13], [14] to the case with finite buffer sizes in the context of wireless communication. The work [11] studies the scheduling issue only, and [12] introduces a joint algorithm for flow control, routing, and scheduling. Packets within a wireless network are classified into different flows based on their source-destination pairs in [11], [12]. In contrast, vehicles within any upstream link within a traffic network form different queues according to their immediate downstream links. Thus, the results in [11], [12] cannot be directly applied to traffic signal control.

This paper considers traffic signal control with finite queue capacities, and the contribution can be summarized below. Firstly, we propose a pressure releasing policy (PRP) to release traffic pressure in an optimal manner. The traffic pressure at each intersection uses knowledge of turning ratios and information of neighboring and ingress queues. PRP does not require knowledge of arrival rates. A set of weights satisfying a given condition is employed in PRP to deal with downstream queue spillover, i.e., blocking of traffic movements from upstream links due to saturation of downstream links [15], and an algorithm is also introduced to generate one possible set of weights. It is worth to emphasize that downstream queue spillover is assumed away by the assumption on infinite queue capacities as in [8]–[10]. Secondly and more importantly, we show that under finite queue capacities and by using Lyapunov methods, PRP can still achieve the closed-loop stability with a reduction on the throughput region defined under infinite queue capacities. The reduction is a function of weights and internal queue capacities, and it is sufficiently small when internal queue capacities are sufficiently large. That is to say that PRP with finite but sufficiently large internal queue capacities can be arbitrarily close to recovering the throughput region defined under infinite queue capacities. Different from the previous
results [8]–[10] for the case with infinite queue capacities
where the stability is defined as the boundedness of all
queues, the stability for the case with finite queue capacities
is defined as the boundedness of ingress queues only, since
internal queues are naturally bounded by their capacities.

The rest of the present paper is organized as follows. Section II and Section III give, respectively, the notation
and the traffic model used in this paper. The throughput
region under infinite queue capacities explored in the existing
work is reviewed in Section IV. In Section V, PRP with
finite queue capacities is introduced, and its stability issue
is investigated in detail. Section VI concludes the paper and
provides future research directions.

II. NOTATION

Throughout this paper, time is slotted with unit length as
t = 0, 1, 2, . . . , and the argument t can be omitted when
there is no ambiguity. The symbol := means “defined as”.
We write |N| and |w| to denote the number of elements in
set N and the absolute value of scalar w, respectively. Use ⊑
or ⊲ to represent the coordinate-wise inequality (or strict
inequality). Let vec(Q) be the vector formed by stacking
the elements of Q into one higher dimensional vector. For
P1, P2, . . . , PM with arbitrary dimensions, define

\[ \text{diag}\{P_1, P_2, \ldots, P_M\} := \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_M \end{bmatrix} \]

where each 0 represents a zero matrix of appropriate dimen-
sion. The expectation operator is denoted by E{·}.

III. TRAFFIC MODEL

A. Topology of Traffic Network

Consider a traffic network with a set of intersections or
nodes given by N := \{n1, n2, . . . , nN\} and |N| = N. Let
a virtual node n0 act as the source of all traffic flows entering
the traffic network and another virtual node nN+1 as the sink
of all traffic flows leaving the traffic network. A link or edge
a within the traffic network can only pass flow from one input
node, denoted by \( \theta_a \), to another output node, denoted by \( \sigma_a \).
We allow for parallel links but exclude the self-loops by
assuming \( \theta_a \neq \sigma_a \) for every link a. We also do not consider
those links passing flow directly from n0 to nN+1. In this
situation, the links within the traffic network can be classified
into three disjoint groups, i.e., entry/ingress links (\( L_1 \)), exit
links (\( L_2 \)), and internal links (\( L \)), based on the property
of their input and output nodes:

\[ a \in \begin{cases} L_1, & \text{if } \theta_a = n_0; \\ L_2, & \text{if } \sigma_a = n_{N+1}; \\ L, & \text{otherwise}. \end{cases} \]

For the convenience of notation, assume a virtual link \( v \)
outside the traffic network passing flow from nN+1 to n0.
Then, the overall interaction topology of the traffic network
and the external world can be conveniently characterized by
a graph G(all) := (N(all), L(all)) with a vertex set N(all) :=
\( N \cup \{n_0, n_{N+1}\} \) and an edge set L(all) := \( L_1 \cup L_2 \cup L \cup \{v\} \).
Moreover, let \( L_1 = |L_1|, L_2 = |L_2|, \) and \( L = |L| \).
Refer to Figure 1 for the graphical depiction of a simple traffic
network with two intersections.

![Graphical depiction of a simple traffic network with intersections n1 and n2.](image)

We use \( \Omega_{\theta_a}^{\text{in}} := \{a|\sigma_a = n\} \) and \( \Omega_{\theta_a}^{\text{out}} := \{a|\theta_a = n\} \) to denote respectively the set of input links and the set of
output links for intersection n ∈ N. It follows that: (i)
\( \Omega_{\theta_a}^{\text{in}}(i) \cap \Omega_{\theta_a}^{\text{in}}(j) = \emptyset \) for \( i \neq j \), and \( \bigcup_{n=1}^N \Omega_{\theta_a}^{\text{in}}(n) = L_1 \cup L \); (ii)
\( \Omega_{\theta_a}^{\text{out}}(i) \cap \Omega_{\theta_a}^{\text{out}}(j) = \emptyset \) for \( i \neq j \), and \( \bigcup_{n=1}^N \Omega_{\theta_a}^{\text{out}}(n) = L_\cup L_2 \). A
phase or stream \((a, b)\) corresponds to the traffic movement
from link a to link b. The set of input links and the set of
output links for any link a ∈ L(all) are defined as \( \Omega_{\theta_a}^{\text{in}} := \{c|(c, a) \text{ is permitted}\} \) and \( \Omega_{\theta_a}^{\text{out}} := \{b|(a, b) \text{ is permitted}\} \),
respectively. Note that specific movements like U-turns may
be prohibited in real traffic systems. It is easy to check that:
(i) \( \Omega_{\theta_a}^{\text{in}}(n) = \{v\} \) for each \( a \in L_1 \), and \( \Omega_{\theta_a}^{\text{in}}(i) = \{v\} \) for each \( a \in L_2 \); (ii)
\( \Omega_{\theta_a}^{\text{in}} \subseteq \Omega_{\theta_a}^{\text{in}} \) and \( \Omega_{\theta_a}^{\text{out}} \subseteq \Omega_{\theta_a}^{\text{out}} \) for all \( a \in L^{\text{all}} \); (iii)
\( \Omega_{\theta_a}^{\text{in}} = L_1 \), and \( \Omega_{\theta_a}^{\text{out}} = L_2 \).

B. Traffic Signals

A set of one or several compatible phases around an
intersection is called a stage. Associate a \( |\Omega_{\theta_a}^{\text{in}}| \times |\Omega_{\theta_a}^{\text{out}}| \)
matrix \( P_{\theta_a} \) with one stage of intersection \( n \in N \), where
\( P_{\theta_a}^{(i) \text{ abs}} = 1 \) or 0 if phase \((a, b)\) with \( a \in \Omega_{\theta_a}^{\text{in}}(i) \) and
\( b \in \Omega_{\theta_a}^{\text{out}}(i) \) is or is not actuated in this stage. The set
of all possible stages for intersection \( n \in N \) is denoted
by \( \mathcal{P} := \{P_{\theta_a}^{(0)}, P_{\theta_a}^{(1)}, P_{\theta_a}^{(2)}, \ldots, P_{\theta_a}^{(M_n)}\} \), where \|\mathcal{P}\| =
\( M_n + 1 \), and \( P_{\theta_a}^{(0)} \) represents the all-red stage which would be
useful in cases of heavy network congestion [16]. Obviously,
\( P_{\theta_a}^{(i) \text{ abs}} = 0 \) for all \( i = 0, 1, \ldots, M_n \), if \( a \in \Omega_{\theta_a}^{\text{in}}(i) \) and
\( b \in \Omega_{\theta_a}^{\text{out}}(i) \). The task of the traffic signal controller is to selec-
t and actuate one stage for every intersection at the
beginning of every time slot. Further denote \( \mathcal{P} := \text{diag}\{P_{\theta_a, \theta_a}, \ldots, P_{\theta_a, \theta_a}\} \). Let \( P_{\theta_a}(t) \in \mathcal{P} \) be the actuated
stage during time slot \( t \) for intersection \( n \in N \), and \( P(t) = \text{diag}\{P_{n_0, n_0}(t), P_{n_1, n_1}(t), \ldots, P_{n_N, n_N}(t)\} \in \mathcal{P} \) of dimension \((L_1 + L) \times (L_1 + L)\) for the collection of actuated stages
for all intersections. Note that an intersection without traffic
lights can be considered as a one-phase intersection, where
the only permitted traffic movement gets the right-of-way all the time.

For the convenience of analysis, we assume that: for any stage \( P_n \in \mathcal{P}_n \) and \( n \in \mathcal{N} \), (i) simultaneous turning movements heading towards the same link are prohibited, i.e., \([P'_{n}]_{ab} = 1\) always implies that \([P_n]_{cb} = 0\) for any \( c \in \Omega^n_{b} \setminus \{a\}\); (ii) it is always admissible not to actuate a phase, i.e., for two 0/1 valued matrices \( P_n, P'_n \), if \( P_n \in \mathcal{P}_n \) and \( P'_n \leq P_n \), then \( P'_n \in \mathcal{P}_n \).

C. Dynamics of Queueing Network

Assume that for any link \( a \), a distinct queue is dedicated to phase \((a, b)\) for each \( b \in \Omega^n_{a} \) (out), and let \( Q_{ab}(t) \) be the length of this queue at the beginning of time slot \( t \). The total queue length of link \( a \) is given by \( Q_a(t) = \sum_{b \in \Omega^n_{a} \text{ (out)}} Q_{ab}(t) \). Refer to Figure 2 for the close-up of link \( a_1 \) in Figure 1.

Fig. 2. Close-up of link \( a_1 \) in Figure 1, illustrating the associated queues. Note that the U-turn \((a_1, c_1)\) is prohibited in this example, and the total queue length of link \( a_1 \) is \( Q_{a_1} = Q_{a_1c_2} + Q_{a_1b_1} + Q_{a_1c_6} \).

There exists a queue capacity \( C_{ab} \) (or \( C_a \)) for each queue \( Q_{ab} \) (or \( Q_a \)). For any exit link \( a \in \mathcal{L}_2 \), only one phase \((a, v)\) is considered. In this case, we can set \( C_{av} = C_a = \infty \) and \( Q_{av}(t) = Q_{a_0}(t) = 0 \) for all \( t \), meaning that all vehicles immediately leave the traffic network once they reach the exit links. For any entry link \( a \in \mathcal{L}_1 \) and \( b \in \Omega^n_{a} \) (out), the ingress queue is assumed to be unlimited, i.e., \( C_{ab} = \infty \), and it updates according to the following queueing dynamics:

\[
Q_{ab}(t + 1) = Q_{ab}(t) + T_{ab}(t)A_a(t) - F_{ab}(t),
\]

where the arrival \( A_a(t) \) is the number of exogenous vehicles arriving at link \( a \) during time slot \( t \), the turning movement \( T_{ab}(t) \) is equal to 1 or 0 if vehicles are or are not routed to queue \( Q_{ab} \) after entering link \( a \) during time slot \( t \), and the flow rate \( F_{ab}(t) \) is the number of vehicles passing from link \( a \) to link \( b \) during time slot \( t \). For any internal link \( a \in \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out), the internal queue evolves as

\[
Q_{ab}(t + 1) = Q_{ab}(t) + T_{ab}(t) \sum_{c \in \Omega^n_{b} (in)} F_{ca}(t) - F_{ab}(t),
\]

where \( \sum_{c \in \Omega^n_{b} (in)} F_{ca}(t) \) represents the number of endogenous vehicles coming from upstream links of link \( a \) during time slot \( t \). Note that (2) can be merged into (3) by setting \( F_{ca}(t) = A_a(t) \) for \( a \in \mathcal{L}_1 \). For brevity, let the collection of queues \( \vec{Q}(t) := \text{vec}(\{Q_{ab}(t)\}) \) be the state of the above queueing network, where \( \vec{Q}(0) \) is the initial state, \( a \in \mathcal{L}_1 \cup \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out).

In this paper, both the arrivals and the turning movements are assumed to random. For each \( a \in \mathcal{L}_1 \), the arrival \( A_a(t) \) is an integer-valued and independent random variable with upper bound \( A_n \) and arrival rate \( A_a := E\{A_a(t)\} \). For each \( a \in \mathcal{L}_1 \cup \mathcal{L} \), the vector of turning movements \( \text{vec}(\{T_{ab}(t)\}) \) with \( b \in \Omega^n_{b} \text{ (out)} \) is a binary independent random vector satisfying \( \sum_{b \in \Omega^n_{b} \text{ (out)}} T_{ab}(t) = 1 \), and we use \( T_{ab} := E\{T_{ab}(t)\} \) to denote the turning ratio of phase \((a, b)\). In addition, \( A_a(t) \) and \( \text{vec}(\{T_{ab}(t)\}) \) are independent from each other. A route of the traffic network is defined as a sequence of traffic movements \( r := \{(a_1, a_2), (a_2, a_3), \ldots, (a_m-1, a_m)\} \) such that \( T_{a_i, a_{i+1}} > 0 \), \( i = 1, 2, \ldots, m-1 \), \( a_1 \in \mathcal{L}_1 \), \( a_m \in \mathcal{L}_2 \) and \( a_2, a_3, \ldots, a_{m-1} \in \mathcal{L} \). Denote the set of all possible routes within the traffic network by \( \mathcal{R} \). In order to ensure that any exogenous vehicle can eventually leave the traffic network, the network topology is assumed to satisfy: (i) the set of exit links \( \mathcal{L}_2 \) is nonempty, i.e., \( \mathcal{L}_2 \neq \emptyset \); (ii) for any link \( a \in \mathcal{L}_1 \cup \mathcal{L} \), there exists at least one route \( r \in \mathcal{R} \) contains \( a \); (iii) there is no self-loop or cyclic route.

Generally, the flow rate from an upstream link to a downstream link during one time slot is determined by the demand (number of vehicles waiting to pass at the upstream link), the supply (reserved queue capacity at the downstream link), as well as the traffic signal. In this paper, the flow rate is given by \( F_{ab}(t) := H_{ab}(t)[P_{a_0}(t)]_{ab} \) for all \( a \in \mathcal{L}_1 \cup \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out). If \( Q_{ab}(t) = 0 \) or \( Q_{bc} = C_{bc} \) for some \( c \in \Omega^n_{b} \) (out), then \( H_{ab}(t) = 0 \); otherwise, \( H_{ab}(t) = 1 \). Based on the above model of queueing network, we can easily derive the following result which states that internal queues within the traffic network never overflow.

**Lemma 1:** For all \( a \in \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out), if the bounded initial state \( \vec{Q}(0) \) satisfies \( Q_{ab}(0) \leq C_{ab} \), then \( Q_{ab}(t) \leq C_{ab} \) for all \( t \).

IV. THROUGHPUT REGION UNDER INFINITE QUEUE CAPACITIES

In this section, we set \( C_{ab} = \infty \) for all \( a \in \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out). Note that \( L = 0 \), e.g., an isolated intersection, is a special case of this section. The existing results reviewed as follows are mainly based on the papers [8], [10].

The stability of the queueing network is given below.

**Definition 4.1:** The queueing network formulated in Section III with state \( \vec{Q}(t) \) is stable, if starting from any bounded initial state \( \vec{Q}(0) \) satisfying \( Q_{ab}(0) \leq C_{ab} \) for all \( a \in \mathcal{L} \) and \( b \in \Omega^n_{a} \) (out), there exists a constant \( V > 0 \) such that

\[
\lim_{t \to \infty} \frac{1}{t+1} \sum_{\tau=0}^{t} \sum_{a \in \mathcal{L}_1 \cup \mathcal{L}} \sum_{b \in \Omega^n_{a} \text{ (out)}} E\{Q_{ab}(\tau)\} \leq V.
\]

The above stability is equivalent to the stability of every individual queue, and \( V \) in (4) provides an upper bound for the limiting time average of the sum of all queue means. The throughput region, denoted by \( \Lambda \), is the closure of the set of all arrival rate vectors, i.e., \( \text{vec}(\{A_a\}) \) with \( a \in \mathcal{L}_1 \), that can be stably supported over the network \( G^{(\text{all})} \), considering all possible traffic signal control algorithms (possibly those with full knowledge of future events) [11]. In the literature, throughput region is also called stability region [17] or network capacity region [18].
Use $\Lambda^o$ to denote the interior of $\Lambda$. Define $\Gamma := \text{Co}\{P|P \in \mathcal{P}\}$, where $\text{Co}(P)$ is the convex hull of $P$. Since $|\mathcal{P}| = \prod_{i=1}^{N} (M_{a_i} + 1)$ is finite, $\Gamma$ is bounded and closed. The next proposition characterizes the throughput region under infinite queue capacities.

**Proposition 4.1:** Suppose all internal links have infinite queue capacities. The throughput region $\Lambda$ of the queueing network consisting all arrival rate vectors $\text{vec}(\{A_a\}, a \in \mathcal{L}_1)$, such that there exist $\{G_{ab}\} \in \Gamma$ and $\{f_{ab}\}$ satisfying

\[
\begin{align*}
  f_{ab} &\geq 0, & \forall a \in \mathcal{L}_1 \cup \mathcal{L}_2, b \in \Omega_a^{(\text{out})}; \\
  f_{ab} &\in \mathcal{L}_1, b \in \Omega_a^{(\text{out})}; \\
  f_{ab} &\in \mathcal{L}_1, b \in \Omega_a^{(\text{out})}; \\
  f_{ab} &\in \mathcal{L}_1, b \in \Omega_a^{(\text{out})}. 
\end{align*}
\]

In addition, if $\text{vec}(\{A_a\}) \in \Lambda^o$, then the last inequality in (5) is strict, i.e., there exists $\epsilon > 0$ such that

\[
f_{ab} \leq G_{ab} - \epsilon. \tag{6}
\]

**V. Pressure Releasing Policy with Finite Queue Capacities**

Next, we turn to consider the general case with finite internal queue capacities, i.e., $L \neq 0$ and $C_{ab} < \infty$ for all $a \in \mathcal{L}_1$ and $b \in \Omega_a^{(\text{out})}$. Denote the set of routes $r \in \mathcal{R}$ satisfying $(a, b) \in r$ by $\mathcal{R}_ab$. For any $a \in \mathcal{L}_1$ and $b \in \Omega_a^{(\text{out})}$, write $\Phi^{(\text{out})}_{ab} := \{(c, d)|c \in \mathcal{L}_2, (c, d) \in r, r \in \mathcal{R}_ab\}$; for $c \in \mathcal{L}_2$ and $d \in \Omega_c^{(\text{out})}$, let $\Phi^{(\text{in})}_{cd} := \{(a, b)|a \in \mathcal{L}_1, (a, b) \in r, r \in \mathcal{R}_cd\}$.

Suppose the state of the queueing network $\bar{Q}(t)$ is available for traffic signal control at the beginning of time $t$. The traffic signal controller under pressure releasing policy (PRP) selects the set of actuated stages $P(t)^* \in \mathcal{P}$ according to

\[
P(t)^* = \arg \max_{P(t) \in \mathcal{P}} \sum_{a \in \mathcal{L}_1 \cup \mathcal{L}_2} \sum_{b \in \Omega_a^{(\text{out})}} W_{ab}(t)[P_{ab}(t)]_{ab}, \tag{7}
\]

where the pressure of phase $(a, b)$ at time $t$ is given by

\[
W_{ab}(t) := w_{ab}(t) - \sum_{c \in \Omega_b^{(\text{out})}} \bar{T}_{bc}w_{bc}(t) \tag{8}
\]

with

\[
w_{ab}(t) := \begin{cases} 
  \alpha_{ab}Q_{ab}(t), & \forall a \in \mathcal{L}_1, b \in \Omega_a^{(\text{out})}; \\
  \frac{\alpha_{ab}\sum_{(a', b') \in \Phi^{(\text{in})}_{ab} Q_{a'b'}(t)}}{c_{ab}} Q_{ab}(t), & \forall a \in \mathcal{L}_1, b \in \Omega_a^{(\text{out})}. 
\end{cases} \tag{9}
\]

The set of weights $\{\alpha_{ab}\}$ with $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$ is selected to satisfy

\[
\alpha_{ab} > 0, \quad \alpha_{ab} \leq \bar{T}_{bc}\alpha_{bc}, \tag{10}
\]

for all $a \in \mathcal{L}_1 \cup \mathcal{L}_2$, $b \in \Omega_a^{(\text{out})}$, and $c \in \Omega_c^{(\text{out})}$. For any traffic network without self-loops and cyclic routes, the weights generation procedure given in Algorithm 1 provides one possible way such that (10) holds.

**Algorithm 1 (Weights Generation)**

Given the set of turning ratios $\{T_{ab}\}$ with $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$, and the set of all possible routes $\mathcal{R}$ with $|\mathcal{R}| = K$.

- **Step 1:** Initialize $i = 1$ and $\alpha_{ab}(0) = 1$ for all $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$.
- **Step 2:** Write the $i$-th route as

\[
r = \{(a_1, a_2), (a_2, a_3), \ldots, (a_{m(i)-1}, a_{m(i)})\}.
\]

For $j = 2, 3, \ldots, m(i) - 1$, set

\[
\alpha_{a_ja_{j+1}}(i) = \max \left\{ \frac{\alpha_{a_ja_{j+1}}(i) - 1}{T_{a_ja_{j+1}}(i)} \right\}.
\]

- **Step 3:** Update $i = i + 1$, and go back to Step 2 until $i = K + 1$.

Output $\alpha_{ab} = \alpha_{ab}(K)$ for all $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$.

Note that if the optimization (7) admits more than one solution, then we can arbitrarily choose one of them with least actuated phases. Since

\[
\sum_{a \in \mathcal{L}_1 \cup \mathcal{L}_2} \sum_{b \in \Omega_a^{(\text{out})}} W_{ab}(t)[P_{ab}(t)]_{ab}
\]

is equivalent to the optimization problem (7) can be equivalently divided into $N$ optimization problems, where $P(t)^* = \text{diag}\{P_{n_1}(t)^*, P_{n_2}(t)^*, \ldots, P_{n_N}(t)^*\}$ and

\[
P_n(t)^* = \arg \max_{P(t) \in \mathcal{P}} \sum_{a \in \Omega_a^{(\text{in})}} \sum_{b \in \Omega_b^{(\text{out})}} W_{ab}(t)[P_{ab}(t)]_{ab} \tag{11}
\]

with $n \in \mathcal{N}$.

**Remark 5.1:** Similarly to back-pressure and max-pressure algorithms [8]–[10], PRP does not require knowledge of arrival rates, but it does use turning ratios to compute traffic pressure in (8) and to generate weights in Algorithm 1. Note that turning ratios can be estimated from historical and/or real-time traffic data; see, e.g., [19], [20].

**Remark 5.2:** As we can see from (9), under finite queue capacities, PRP at one intersection would use information of ingress queues in addition to those queues adjacent to the intersection. Under infinite queue capacities, one can set $w_{ab} = \alpha_{ab}Q_{ab}(t)$ for all $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$ to replace (9), then PRP is reduced to weighted max-pressure control [10] and signal controller at each intersection only requires information of neighboring queues.

The next three lemmas list several properties of PRP, whose proofs are omitted for brevity.

**Lemma 2:** For PRP (7) or (11) with the pressure in (8) and the weights satisfying (10), if $H_{ab}(t) = 0$ for any $a \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $b \in \Omega_a^{(\text{out})}$, then $[P_{ab}(t)]_{ab} = 0$.

**Remark 5.3:** The assumption on infinite queue capacities assumes away downstream queue spillover, i.e., blocking of...
traffic movements from upstream links due to saturation of downstream links. For the case with finite queue capacities, the set of weights satisfying (10) is adopted in PRP to handle downstream queue spillover, and it also plays an important role in establishing the property of PRP given in Lemma 2. In addition, the set of weights satisfying (10) is generally not unique, and Algorithm 1 only produces one possible set. Therefore, we can assign larger weights under the condition (10) to those phases with higher priority, e.g., phases dedicated to buses, without affecting the validity of the main results in this paper.

**Lemma 3:** For PRP (7) or (11) with the pressure in (8) and the weights satisfying (10), there exists \( \{f_{ab}\} \) as defined in Proposition 4.1 such that
\[
\sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \alpha_{ab} Q_{ab}(t) T_{ab} A_a = \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} W_{ab}(t) f_{ab}.
\]

**Lemma 4:** For PRP (7) or (11) with the pressure in (8) and the weights satisfying (10), there exists \( \eta > 0 \) such that
\[
\sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \sum_{(c,d) \in \Phi_{ab}} \alpha_{ab} T_{ab} A_a C_{cd}.
\]

Define
\[
D_{ab} := \max \{ \hat{A}_{ab}^2, 1 \}, \forall a \in L_1, b \in \Omega_a^{(out)}, \quad (12)
\]
and
\[
D := \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \alpha_{ab} D_{ab}
\]
\[
+ \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \sum_{(c,d) \in \Phi_{ab}} \alpha_{ab} T_{ab} A_a C_{cd}. \quad (13)
\]

Further let
\[
\hat{D}_{ab} := \sum_{(c,d) \in \Phi_{ab}} \frac{\alpha_{cd}}{C_{cd}}, \quad \hat{D} := \max_{a \in L_1, b \in \Omega_a^{(out)}} \{ \hat{D}_{ab} \}. \quad (14)
\]

Based on Lemma 1, we have \( Q_{ab}(t) \leq C_{ab} \) for all \( a \in L \) and \( b \in \Omega_a^{(out)} \), and thus the definition of stability given in (4) can be simplified into
\[
\lim_{t \to \infty} \frac{1}{t+1} \sum_{\tau=0}^{t} \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \mathbb{E}\{ Q_{ab}(\tau) \} \leq V. \quad (15)
\]

The theorem below, as the main result of this paper, provides the stability condition for PRP under finite queue capacities.

**Theorem 5.1:** Suppose that all internal links have finite queue capacities. If \( \text{vec}(\{A_a\}) \in \Lambda^o \) and \( \epsilon \) in (6) satisfies
\[
\epsilon > \frac{\hat{D}}{2\eta} \quad (16)
\]
\[
\text{Proof:} \quad \text{For any} \quad a \in L_1 \quad \text{and} \quad b \in \Omega_a^{(out)} \quad \text{it follows from (2) that}
\]
\[
= Q_{ab}(t)^2 + \left[ T_{ab}(t) A_a(t) - F_{ab}(t) \right]^2
\]
\[
+ 2 Q_{ab}(t) \left[ T_{ab}(t) A_a(t) - F_{ab}(t) \right]
\]
\[
\leq Q_{ab}(t)^2 + D_{ab} + 2 Q_{ab}(t) \left[ T_{ab}(t) A_a(t) - F_{ab}(t) \right],
\]
where \( D_{ab} \) is defined in (12).

For \( a \in L_1, b \in \Omega_a^{(out)} \) and \( c \in L, d \in \Omega_c^{(out)} \), it follows from (2) and (3) that
\[
Q_{ab}(t + 1) Q_{cd}(t + 1)^2
\]
\[
= [Q_{ab}(t) + T_{ab}(t) A_a(t) - F_{ab}(t)] Q_{cd}(t + 1)^2
\]
\[
= Q_{ab}(t) \left[ Q_{cd}(t) + T_{cd}(t) \sum_{e \in \Omega_e^{(in)}} F_{ec}(t) - F_{cd}(t) \right]^2
\]
\[
+ \left[ T_{ab}(t) A_a(t) - F_{ab}(t) \right] Q_{cd}(t + 1)^2
\]
\[
\leq Q_{ab}(t) Q_{cd}(t)^2 + Q_{ab}(t) + T_{ab}(t) A_a(t) C_{cd}^2
\]
\[
+ 2 Q_{ab}(t) Q_{cd}(t) \left[ T_{cd}(t) \sum_{e \in \Omega_e^{(in)}} F_{ec}(t) - F_{cd}(t) \right],
\]
where
\[
\left[ T_{cd}(t) \sum_{e \in \Omega_e^{(in)}} F_{ec}(t) - F_{cd}(t) \right]^2 \leq 1,
\]
\[
Q_{cd}(t + 1) \leq C_{cd}
\]
are used in the last inequality.

Construct a Lyapunov function as
\[
Y(\bar{Q}(t)) = \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \alpha_{ab} Q_{ab}(t)^2
\]
\[
+ \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \sum_{(c,d) \in \Phi_{ab}} \frac{\alpha_{cd}}{C_{cd}} Q_{ab}(t) Q_{cd}(t)^2. \quad (17)
\]
Note that a simpler form of the above Lyapunov function is used in [11], [12]. Then, it holds that
\[
\mathbb{E}\{ Y(\bar{Q}(t + 1)) - Y(\bar{Q}(t)) | \bar{Q}(t) \}
\]
\[
\leq D + 2 \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \alpha_{ab} Q_{ab}(t) \left[ \bar{T}_{ab} A_a - F_{ab}(t) \right]
\]
\[
+ \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \sum_{(c,d) \in \Phi_{ab}} \frac{\alpha_{cd}}{C_{cd}} Q_{ab}(t)
\]
\[
+ 2 \sum_{a \in L_1} \sum_{b \in \Omega_a^{(out)}} \sum_{(c,d) \in \Phi_{ab}} \frac{\alpha_{cd}}{C_{cd}} Q_{ab}(t) Q_{cd}(t)
\]
\[
\times \left[ \bar{T}_{cd} \sum_{e \in \Omega_e^{(in)}} F_{ec}(t) - F_{cd}(t) \right]
\]
\[
:= D + \alpha(t),
\]
where $D$ is defined in (13). Rearranging $\alpha$ according to $F_{ab}$ yields

$$
\alpha(t) = 2 \sum_{a \in L_1} \sum_{b \in \Omega(\text{out})} Q_{ab}(t) [\alpha_{ab} \bar{T}_{ab} \bar{\Lambda}_a + \frac{1}{2} \bar{D}_{ab}] 
- 2 \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} W_{ab}(t) H_{ab}(t) |P_{\sigma_a}(t)|_{ab}.
$$

Lemma 2 implies that $H_{ab}(t) |P_{\sigma_a}(t)|_{ab} = |P_{\sigma_a}(t)|_{ab}$.

Then, according to Lemma 3, it holds that

$$
\alpha(t) = \sum_{a \in L_1} \sum_{b \in \Omega(\text{out})} \hat{D}_{ab} Q_{ab}(t)
+ 2 \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} (f_{ab} - |P_{\sigma_a}(t)|_{ab}) W_{ab}(t)
$$

Note that PRP maximizes the term

$$
\sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} |P_{\sigma_a}(t)|_{ab} W_{ab}(t)
$$

for every time slot $t$. Therefore, if $\text{vec}\{\bar{\Lambda}_a\} \in \Lambda^e$, PRP is used, and the condition (16) is true, then it follows from Lemma 4 that

$$
E\{Y(\bar{Q}(t + 1)) - Y(\bar{Q}(t))\bar{Q}(t)\leq D + \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} \hat{D}_{ab} Q_{ab}(t) - 2\varepsilon \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} |W_{ab}(t)|
\leq D - (2\epsilon\eta - \bar{D}) \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} Q_{ab}(t),
$$

which further implies

$$
\lim_{t \to \infty} \frac{1}{t + 1} \sum_{\tau = 0}^{t} \sum_{a \in L_1 \cup \mathcal{L}} \sum_{b \in \Omega(\text{out})} Q_{ab}(\tau)
\leq \lim_{t \to \infty} \frac{Y(\bar{Q}(0)) - Y(\bar{Q}(t + 1)) + D}{(2\varepsilon\eta - \bar{D})(t + 1)} + \frac{D}{2\varepsilon\eta - \bar{D}}
\leq \lim_{t \to \infty} \frac{Y(\bar{Q}(0))}{(2\varepsilon\eta - \bar{D})(t + 1)} + \frac{D}{2\varepsilon\eta - \bar{D}} = \frac{D}{2\varepsilon\eta - \bar{D}}.
$$

which completes the proof. \hfill \Box

Remark 5.4: Note that $\frac{D}{2\eta}$ in (16) characterizes the reduction of PRP with finite queue capacities on the through region $\Lambda$ defined under infinite queue capacities. It is observed that if internal queue capacities are sufficiently large, then $D$ in (14) and $\frac{D}{2\eta}$ in (16) are sufficiently small, indicating that PRP with finite but sufficiently large internal queue capacities can be arbitrarily close to recovering the throughput region $\Lambda$.

VI. CONCLUSIONS

In this paper, PRP has been proposed for traffic signal control under finite queue capacities. It has been shown that under finite queue capacities, PRP can still achieve the closed-loop stability with a reduction on the throughput region defined under infinite queue capacities, where the reduction is sufficiently small if internal queue capacities are sufficiently large.

Our future work includes: (i) optimization over other performance indexes, e.g., average speed, average delay per vehicle, average number of stops per vehicle, without affecting network throughput; (ii) analysis of overload behavior at the heavy traffic regime, etc.

REFERENCES


